

ANNIHILATION OF TORSION IN HOMOLOGY OF FINITE m -AQ QUANDLES

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ABSTRACT. It is a classical result in reduced homology of finite groups that the order of a group annihilates its homology. Similarly, we have proved that the torsion subgroup of rack and quandle homology of a finite quasigroup quandle is annihilated by its order. However, it does not hold for connected quandles in general. In this paper, we define an m -almost quasigroup (m -AQ) quandle which is a generalization of a quasigroup quandle and study annihilation of torsion in its rack and quandle homology groups.

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1. INTRODUCTION

A *quandle* is an algebraic structure with a set X and a binary operation $*$: $X \times X \rightarrow X$ satisfying the following axioms:

- (1) (idempotency) $a * a = a$ for any $a \in X$,
- (2) (invertibility) for each $b \in X$, $*_b : X \rightarrow X$ given by $*_b(x) = x * b$ is invertible (the inverse is denoted by $\overline{*}_b$),
- (3) (right self-distributivity) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$.

A set X with a binary operation $*$: $X \times X \rightarrow X$ that satisfies the right self-distributive property and invertibility is called a *rack*.

Rack and quandle theory is motivated by knot theory. The axioms of a quandle correspond to the three Reidemeister moves [Joy, Matv]. When we color an oriented knot diagram by a given magma $(X; *)$ consisting of a finite set X and a binary operation $*$: $X \times X \rightarrow X$ with the convention of Figure 1.1, we need the above conditions (1), (2), and (3) to be satisfied in order that Reidemeister moves preserve the number of the allowed colorings. Quandles can be used for classifying classical knots [Joy, Matv].

The followings are some examples of quandles:

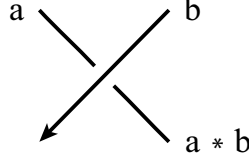


Figure 1.1; Quandle coloring

- Example 1.1.** (1) Let X be a set with a binary operation $a * b = a$ for any $a, b \in X$. Then X is a quandle called a *trivial quandle*.
- (2) A group G with the conjugate operation $g * h = h^{-1}gh$ forms a quandle structure called a *conjugate quandle*. More generally, any subset of G which is closed under conjugation is a subquandle of a conjugate quandle G .
- (3) [Tak] If G is an abelian group, we define a quandle called a *Takasaki quandle* (or *kei*) of G , denoted by $T(G)$, by taking $a * b = 2b - a$. Specially, if $G = \mathbb{Z}_n$, then we denote $T(\mathbb{Z}_n)$ by R_n called a *dihedral quandle*.
- (4) Let M be a module over the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. Then a quandle M with the operation $a * b = ta + (1 - t)b$ is said to be an *Alexander quandle*.

Elementary quandle theory has been developed in a way similar to basic group theory. A function $h : X \rightarrow Y$ between two quandles $(X; *)$ and $(Y; *')$ is said to be a *quandle homomorphism* if $h(a * b) = h(a) *' h(b)$ for any $a, b \in X$. If a quandle homomorphism is invertible, then we call it a *quandle isomorphism*. A *quandle automorphism* is a quandle isomorphism from a quandle X onto itself.

Recall that $*$ has an inverse binary operation $\bar{*}$, that is $(a * b)\bar{*}b = a = (a\bar{*}b) * b$ for any $a, b \in X$. A *subquandle* of a quandle X is a subset S of X which is closed under $*$ and $\bar{*}$ operations.

Observation 1.2. If $(X; *)$ is a finite quandle, then $S \subset X$ is a subquandle if it is closed under $*$ operation. It is the case because for any $a, b \in X$ the sequence $a *^k b$ where $a *^k b = (\cdots ((a * b) * b) * \cdots) * b$ has to have repetitions, say $a *^m b = a *^n b$ for $0 \leq m < n$, so $a *^{n-m} b = a$. That is, $a\bar{*}b = a *^{n-m-1} b$.

A quandle X is said to be a *quasigroup quandle* (or *Latin quandle*) if it satisfies the quasigroup property, i.e. for any $a, b \in X$, the equation $a * x = b$ has a unique solution x . For each $a \in X$, the map $*_a : X \rightarrow X$ defined by $*_a(x) = x * a$ is a quandle automorphism. A subgroup of the quandle automorphism group $\text{Aut}(X)$ of a quandle X generated by $*_a$ for every $a \in X$ is called a *quandle inner automorphism group* denoted by $\text{Inn}(X)$. If the action of $\text{Inn}(X)$ on a quandle X is transitive¹, then we call X a *connected quandle*. Note that if a quandle X has the quasigroup property, then it is a connected quandle. The converse, however, does not hold in general (see Example 1.3 (3)).

- Example 1.3.** (1) Dihedral quandles R_n of odd order are quasigroup quandles.
- (2) An Alexander quandle M is a quasigroup quandle if and only if $1 - T$ is invertible.

¹It means that for any a and b in a quandle X there are $x_1, \dots, x_n \in X$ so that $(\cdots ((a *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) *^{\varepsilon_3} \cdots) *^{\varepsilon_n} x_n = b$ where $*^{\varepsilon} = *$ or $\bar{*}$. For a finite quandle X , by Observation 1.2, we can find $y_1, \dots, y_k \in X$ so that $(\cdots ((a * y_1) * y_2) * \cdots) * y_k = b$.

- (3) The quandle $QS(6)$ (or Rig quandle² $Q(6, 2)$), which is the orbit of 4-cycles in the symmetric group S_4 of order 24 with the conjugate operation, is a connected quandle but not a quasigroup quandle (see [CKS-1] and [CKS-2]).

Rack homology theory was introduced by Fenn, Rourke, and Sanderson [FRS], and it was modified by Carter, Jelsovsky, Kamada, Langford, and Saito [CJKLS] to define a (quandle) cocycle knot invariant. We recall the definition of rack and quandle homology based on [CKS-2], starting from the rack chain complex.

Definition 1.4. Let $C_n^R(X)$ be the free abelian group generated by n -tuples (x_1, \dots, x_n) of elements of a rack X , i.e. $C_n^R(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. We define a boundary homomorphism $\partial_n : C_n^R(X) \rightarrow C_{n-1}^R(X)$ by

$$\partial_n(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^i (d_i^{(*0)} - d_i^{(*)})(x_1, \dots, x_n)$$

where $d_i^{(*0)}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and

$$d_i^{(*)}(x_1, \dots, x_n) = (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n).$$

The face maps $d_i^{(*)}$ and $d_i^{(*0)}$ are illustrated in Figure 1.2.

Then $(C_n^R(X), \partial_n)$ is said to be a *rack chain complex* of X .

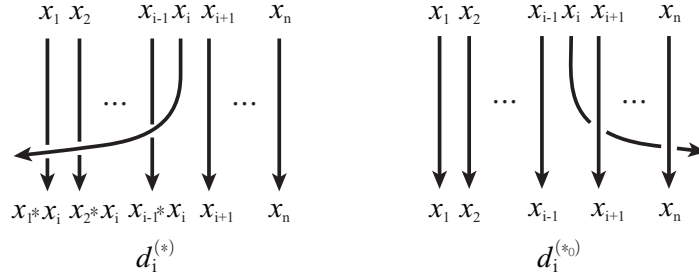


Figure 1.2; Graphical descriptions of face maps $d_i^{(*)}$ and $d_i^{(*0)}$

Definition 1.5. For a quandle X , we consider the subgroup $C_n^D(X)$ of $C_n^R(X)$ generated by n -tuples (x_1, \dots, x_n) of elements of X with $x_i = x_{i+1}$ for some $i = 1, \dots, n-1$. Then $(C_n^D(X), \partial_n)$ is the subchain complex of a rack chain complex $(C_n^R(X), \partial_n)$ and it is called the *degenerate chain complex* of X . Then we take the quotient chain complex $(C_n^Q(X), \partial_n) = (C_n^R(X)/C_n^D(X), \partial_n)$ and call it the *quandle chain complex*.

Definition 1.6. For an abelian group G , we define the chain complex $C_*^W(X; G) = C_*^W(X) \otimes G$ with $\partial = \partial \otimes \text{Id}$ for $W=R, D$, and Q . Then the *n th rack, degenerate, and quandle homology groups of a quandle X with coefficient group G* are respectively defined as

$$H_n^W(X; G) = H_n(C_*^W(X; G)) \text{ for } W=R, D, \text{ and } Q.$$

²L. Vendramin obtained a list of all connected quandles of order less than 48 by using GAP, and they can be found in the GAP package Rig [Ven]. There are 431 connected quandles of order less than 36 [CSV]. These quandles are called *Rig quandles*, and we denote i -th quandle of order n in the list of Rig quandles by $Q(n, i)$.

For any finite rack or quandle, the free parts of rack, degenerate, and quandle homology groups have been completely obtained in [CJKS, E-G, L-N].

Theorem 1.7. [CJKS, E-G, L-N] *Let \mathcal{O} be the set of orbits of a rack X with respect to the action of X on itself by right multiplication. Then*

- (1) $\text{rank}H_n^R(X) = |\mathcal{O}|^n$ for a finite rack X ,
- (2) $\text{rank}H_n^Q(X) = |\mathcal{O}|(|\mathcal{O}| - 1)^{n-1}$ for a finite quandle X ,
- (3) $\text{rank}H_n^D(X) = |\mathcal{O}|^n - |\mathcal{O}|(|\mathcal{O}| - 1)^{n-1}$ for a finite quandle X .

For every finite connected quandle X , by Theorem 1.7 we have $\text{rank}H_n^R(X) = 1$ for all n , $\text{rank}H_1^Q(X) = 1$, and $\text{rank}H_n^Q(X) = 0$ if $n \geq 2$. Annihilation of torsion subgroups of rack and quandle homology of quasigroup quandles was addressed in [P-Y]³:

Theorem 1.8. [P-Y] *Let Q be a finite quasigroup quandle. Then the torsion subgroup of $H_n^R(Q)$ is annihilated by $|Q|$.*

Corollary 1.9. [P-Y] *The reduced quandle homology of a finite quasigroup quandle is annihilated by its order, i.e. $|Q|\tilde{H}_n^Q(Q) = 0$.*

We can generalize the Theorem 1.8 to finite quasigroup racks.

Corollary 1.10. *Let X be a finite quasigroup rack. Then the torsion subgroup of $H_n^R(X)$ is annihilated by $|X|$.*

Proof. It is well known that in a rack, b and $b * b$ are functionally the same. Namely for any $a \in X$ we have

$$a * b = ((a \bar{*} b) * b) * b = a * (b * b).$$

From the quasigroup property we know that if $a * x = a * y$ then $x = y$ thus in a quasigroup rack $b = b * b$ and the rack is a quandle. \square

The 6-elements quandle $QS(6)$ (or Rig quandle $Q(6, 2)$, see Table 1 and Example 1.3 (3)) is a connected quandle, but it is not a quasigroup quandle and $H_3^Q(QS(6)) = \mathbb{Z}_{24}$ (see [CKS-1]). Thus 6 does not annihilate $\text{tor}H_3(QS(6))$; this shows that Theorem 1.8 and Corollary 1.9 do not hold for connected quandles in general. We show that the torsion subgroup of rack and quandle homology of $QS(6)$ is annihilated by the order of its quandle inner automorphism group (24 in this case) in Theorem 3.2.

TABLE 1. A quandle $QS(6)$

$*$	1	2	3	4	5	6
1	1	1	6	5	3	4
2	2	2	5	6	4	3
3	5	6	3	3	2	1
4	6	5	4	4	1	2
5	4	3	1	2	5	5
6	3	4	2	1	6	6

³Annihilation of torsion of homology of R_3 by 3 was proven in [N-P-1]. See [P-Y] for the history of annihilation problem.

2. m -ALMOST QUASIGROUP (m -AQ) QUANDLES

An m -almost quasigroup quandle (m -AQ quandle, in short) is a generalization of a quasigroup quandle. A formal definition follows.

Definition 2.1. A quandle X is said to be m -almost quasigroup if it satisfies the following conditions:

- (1) for each $a \in X$, the stabilizer set $S_a = \{x \in X | a * x = a\}$ has order m , say $S_a = \{a = a^{(1)}, a^{(2)}, \dots, a^{(m)}\}$,
- (2) for any $b \in X \setminus S_a$, the equation $a * x = b$ has a unique solution.

If $m = 1$ above, then X is a quasigroup quandle. Note that if X is finite and $m \geq 2$, then the equation $a * x = a^{(k)}$ has no solution if $2 \leq k \leq m$.

The following are some examples of m -almost quasigroup quandles:

- Example 2.2.**
- (1) Every finite trivial quandle X is an $|X|$ -almost quasigroup quandle.
 - (2) The quandle $QS(6)$ (or Rig quandle $Q(6, 2)$) is a 2-almost quasigroup quandle.
 - (3) A quandle composed of 2-cycles in the symmetric group S_n and the conjugate operation is an $(\binom{n-2}{2} + 1)$ -almost quasigroup quandle if $n \geq 2$ (for 2-cycles in S_n , $(i, j) * (k, l) = (i, j)$ if and only if $\{i, j\} = \{k, l\}$ or $\{i, j\} \cap \{k, l\} = \emptyset$). In particular, Rig quandles $Q(6, 1)$, $Q(10, 1)$, $Q(15, 7)$, $Q(21, 9)$, and $Q(28, 13)$ are the cases of $n = 4$, $n = 5$, $n = 6$, $n = 7$, and $n = 8$, respectively.
 - (4) The Rig quandle $Q(15, 2)$ which is composed of elements of S_5 with cycle partition $(2, 2, 1)$ is a 3-almost quasigroup quandle.

We check some basic properties of m -almost quasigroup quandles:

Lemma 2.3. Suppose that X is an m -almost quasigroup quandle.

- (1) For each $a \in X$, the stabilizer set S_a is a subquandle of X .
- (2) For any $a, b \in X$, $S_a * b = S_{a*b}$ where $S_a * b = \{a * b, a^{(2)} * b, \dots, a^{(m)} * b\}$.
- (3) If X is finite and $m \leq 3$, then every stabilizer set is a trivial subquandle of X .
- (4) If X is nontrivial and every stabilizer set is a trivial subquandle of X , then X is connected.

Proof. (1) For $a \in X$, we consider the stabilizer set $S_a = \{a = a^{(1)}, a^{(2)}, \dots, a^{(m)}\}$. Then $a = a * a^{(j)} = (a * a^{(i)}) * a^{(j)} = (a * a^{(j)}) * (a^{(i)} * a^{(j)}) = a * (a^{(i)} * a^{(j)})$ so that $a^{(i)} * a^{(j)} \in S_a$ for any $1 \leq i, j \leq m$. Moreover, $a * a^{(j)} = a$ implies that $a = a \bar{*} a^{(j)}$ so that $a = (a * a^{(i)}) \bar{*} a^{(j)} = (a \bar{*} a^{(j)}) * (a^{(i)} \bar{*} a^{(j)}) = a * (a^{(i)} \bar{*} a^{(j)})$. Therefore we similarly have $a^{(i)} \bar{*} a^{(j)} \in S_a$ for any $1 \leq i, j \leq m$. That is, S_a is closed under both operations $*$ and $\bar{*}$, hence S_a is a subquandle of X .

(2) For any $i \in \{1, 2, \dots, m\}$, $(a * b) * (a^{(i)} * b) = (a * a^{(i)}) * b = a * b$, and this implies $S_a * b \subset S_{a*b}$. Since $*_b$ is bijective, $|S_a * b| = m = |S_{a*b}|$ therefore $S_a * b = S_{a*b}$.

(3) Let $a \in X$, then the stabilizer set S_a is a subquandle of X by Lemma 2.3 (1). If $m \leq 2$, then S_a is clearly trivial, so we are done. Assume that $m = 3$ and denote $S_a = \{a, b, c\}$. Since $a * c = a$ and $c * c = c$, by the invertibility condition of a quandle we have $b * c = b$, i.e. $c \in S_b$. Then $b * a$ should be b or c because S_a is closed under $*$ and $a * a = a$. Since X is finite, the second axiom of the definition of an m -almost quasigroup quandle implies that for any $y \in S_b$ the equation $b * x = y$ has no solution. Therefore, the equation $b * x = c$ has no solution, so $b * a = b$. Then $c * a = c$ and $c * b = c$ by the invertibility condition of a quandle, therefore S_a is a trivial subquandle of X .

(4) Since X is a nontrivial m -almost quasigroup quandle, $m < |X|$. Let $a, b \in X$. If b is not contained in S_a , then there exists a unique solution $y \in X$ such that $a * y = b$ by the second axiom

of the definition of an m -almost quasigroup quandle. Suppose that $b \in S_a$. Then the triviality of S_a implies that $S_a = S_b$. Since $m < |X|$, there is an element $c \in X$ such that $c \in X \setminus S_a$. Then we can find a unique element $y_1 \in X$ so that $a * y_1 = c$. We easily check that $b \in X \setminus S_c$ (because if $b \in S_c$, then since S_c is trivial, we have the equation $b * c = b$, i.e. $c \in S_b = S_a$, and this contradicts $c \in X \setminus S_a$). Then there is a unique element $y_2 \in X$ so that $c * y_2 = b$, and then we have $(a * y_1) * y_2 = c * y_2 = b$. Therefore the quandle X is connected. \square

Notice that the stabilizer set $S_a = \{a = a^{(1)}, a^{(2)}, \dots, a^{(m)}\}$ is a trivial subquandle of a quandle X if and only if $S_a = S_{a^{(k)}}$ for any k . In general the equality $S_a = S_{a^{(k)}}$ does not have to hold; see the Table 2.

Carter, Elhamdadi, Nikiforou, and Saito [CENS] introduced an abelian extension theory of quandles, and a generalization to extensions with a dynamical cocycle was defined by Andruskiewitsch and Graña [AG]. We review the definition of quandle extension by a dynamical cocycle after [CHNS].

Definition 2.4. [AG, CHNS] Let X be a quandle and S be a non-empty set. Let $\alpha : X \times X \rightarrow \text{Fun}(S \times S, S) = S^{S \times S}$ be a function, so that for $a, b \in X$ and $s, t \in S$ we have $\alpha_{a,b}(s, t) \in S$. Then $S \times X$ is a quandle by the operation $(s, a) * (t, b) = (\alpha_{a,b}(s, t), a * b)$, where $a * b$ denotes the quandle operation in X , if and only if α satisfies the following conditions:

- (1) $\alpha_{a,a}(s, s) = s$ for all $a \in X$ and $s \in S$,
- (2) $\alpha_{a,b}(-, t) : S \rightarrow S$ is a bijection for all $a, b \in X$ and for all $t \in S$,
- (3) $\alpha_{a*b,c}(\alpha_{a,b}(s, t), u) = \alpha_{a*c,b*c}(\alpha_{a,c}(s, u), \alpha_{b,c}(t, u))$ for all $a, b, c \in X$ and $s, t, u \in S$.

Such a function α is called a *dynamical quandle cocycle*. The quandle constructed above is denoted by $S \times_\alpha X$, and is called the *extension* of X by a dynamical cocycle α .

Unlike quasigroup quandles, m -almost quasigroup quandles are not connected in general. We obtain interesting examples of non-connected m -almost quasigroup quandles from quandle extensions of quasigroup quandles and trivial quandles.

Example 2.5. Let $(X; *)$ be a finite quasigroup quandle and T_n the trivial quandle of order n . Then the dynamical quandle cocycle extension $X \times_\alpha T_n$ is an $((n-1)|X| + 1)$ -almost quasigroup quandle (but not a connected quandle if $n \geq 2$) where the dynamical quandle cocycle $\alpha : T_n \times T_n \rightarrow X^{X \times X}$ is defined by $\alpha_{a,b}(s, t) = s * t$ if $a = b$ and $\alpha_{a,b}(s, t) = s$ if $a \neq b$.

TABLE 2. A 4-almost quasigroup quandle which is not connected; $R_3 \times_\alpha T_2$

$*$	1	2	3	4	5	6
1	1	3	2	1	1	1
2	3	2	1	2	2	2
3	2	1	3	3	3	3
4	4	4	4	4	6	5
5	5	5	5	6	5	4
6	6	6	6	5	4	6

We computed that $H_2^R(R_3 \times_\alpha T_2) = \mathbb{Z}^4$, $H_3^R(R_3 \times_\alpha T_2) = \mathbb{Z}^8 \oplus \mathbb{Z}_3^2$, $H_4^R(R_3 \times_\alpha T_2) = \mathbb{Z}^{16} \oplus \mathbb{Z}_3^8$, $H_2^Q(R_3 \times_\alpha T_2) = \mathbb{Z}^2$, $H_3^Q(R_3 \times_\alpha T_2) = \mathbb{Z}^2 \oplus \mathbb{Z}_3^2$, and $H_4^Q(R_3 \times_\alpha T_2) = \mathbb{Z}^2 \oplus \mathbb{Z}_3^6$.

3. ANNIHILATION OF RACK AND QUANDLE HOMOLOGY GROUPS OF m -ALMOST QUASIGROUP QUANDLES

Our main result of the paper is Theorem 3.2. We start from the important preparatory Lemma 3.1.

Let X be an m -almost quasigroup quandle and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. We consider two chain maps $g_1^j, g_2^j : C_n^R(X) \rightarrow C_n^R(X)$ given by

$$g_1^j(\mathbf{x}) = \sum_{k=1}^m (x_j^{(k)}, \dots, x_j^{(k)}, x_j, x_{j+1}, \dots, x_n),$$

$$g_2^j(\mathbf{x}) = \sum_{k=1}^m (x_j^{(k)}, \dots, x_j^{(k)}, x_j^{(k)}, x_{j+1}, \dots, x_n)$$

for $1 \leq j \leq n$. Note that the chain map g_1^1 is equal to $m\text{Id}$.

Lemma 3.1. *Let X be a finite m -almost quasigroup quandle. Suppose that for each $a \in X$ the stabilizer set S_a is a trivial subquandle of X . Then the chain maps $(|X| - m)g_1^j$ and $(|X| - m)g_2^j$ are chain homotopic for each $1 \leq j \leq n$.*

Proof. We define a chain map $g_0^j : C_n^R(X) \rightarrow C_n^R(X)$ by

$$g_0^j(\mathbf{x}) = \sum_{k=1}^m (x_j, \dots, x_j, x_j^{(k)}, x_{j+1}, \dots, x_n)$$

for $1 \leq j \leq n$. Notice that $g_0^1 = g_2^1$.

We consider a chain homotopy $G_n^j : C_n^R(X) \rightarrow C_{n+1}^R(X)$ given by for $1 \leq j \leq n$

$$G_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} \{ (x_j^{(k)}, \dots, x_j^{(k)}, x_j, y, x_{j+1}, \dots, x_n) - (x_j, \dots, x_j, x_j^{(k)}, y, x_{j+1}, \dots, x_n) \}.$$

If $i \leq j - 1$, then by the idempotence condition of a quandle we have

$$d_i^{(*)} G_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} \{ (x_j^{(k)}, \dots, x_j^{(k)}, x_j, y, x_{j+1}, \dots, x_n) - (x_j, \dots, x_j, x_j^{(k)}, y, x_{j+1}, \dots, x_n) \}$$

so that the formula above does not depend on the binary operation $*$, therefore $(d_i^{(*)0} - d_i^{(*)})G_n^j = 0$.

If $i = j$, then the formula again does not depend on the binary operation $*$ because the stabilizer set S_{x_j} is trivial

$$d_i^{(*)} G_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} \{ (x_j^{(k)}, \dots, x_j^{(k)}, y, x_{j+1}, \dots, x_n) - (x_j, \dots, x_j, y, x_{j+1}, \dots, x_n) \}$$

thus $(d_i^{(*)0} - d_i^{(*)})G_n^j = 0$.

If $i = j + 1$, then since the stabilizer set S_{x_j} is trivial and X satisfies the m -almost quasigroup property, we have $\sum_{y \in X} (x_j^{(k)} * y) = m(x_j^{(k)}) + \sum_{y \in X \setminus S_{x_j}} (x_j^{(k)} * y) = m(x_j^{(k)}) + \sum_{y \in X \setminus S_{x_j}} (y)$ and therefore

$$\begin{aligned} d_i^{(*)} G_n^j(\mathbf{x}) &= \sum_{k=1}^m \{m(x_j^{(k)}, \dots, x_j^{(k)}, x_j, x_{j+1}, \dots, x_n) + \sum_{y \in X \setminus S_{x_j}} (x_j^{(k)} * y, \dots, x_j^{(k)} * y, x_j * y, x_{j+1}, \dots, x_n) \\ &\quad - m(x_j, \dots, x_j, x_j^{(k)}, x_{j+1}, \dots, x_n) - \sum_{y \in X \setminus S_{x_j}} (x_j * y, \dots, x_j * y, x_j^{(k)} * y, x_{j+1}, \dots, x_n)\} \\ &= m \sum_{k=1}^m \{(x_j^{(k)}, \dots, x_j^{(k)}, x_j, x_{j+1}, \dots, x_n) - (x_j, \dots, x_j, x_j^{(k)}, x_{j+1}, \dots, x_n)\}. \end{aligned}$$

The part corresponding to the sum over $y \in X \setminus S_{x_j}$ cancels out because for any $x_j, x_j^{(k)} \in S_{x_j}$, and $y \in X \setminus S_{x_j}$, there are unique $l \in \{1, \dots, m\}$ and $z \in X \setminus S_{x_j}$ so that $(x_j^{(k)} * y, x_j * y) = (x_j * z, x_j^{(l)} * z)$.

Then we have

$$(d_i^{(*)0} - d_i^{(*)}) G_n^j(\mathbf{x}) = (|X| - m) \sum_{k=1}^m \{(x_j^{(k)}, \dots, x_j^{(k)}, x_j, x_{j+1}, \dots, x_n) - (x_j, \dots, x_j, x_j^{(k)}, x_{j+1}, \dots, x_n)\}.$$

Lastly, if $j + 2 \leq i \leq n + 1$, then by the invertibility condition of a quandle we have $\sum_{y \in X} (y * x_{i-1}) = \sum_{y \in X} (y)$ and therefore

$$\begin{aligned} d_i^{(*)} G_n^j(\mathbf{x}) &= \sum_{k=1}^m \sum_{y \in X} \{(x_j^{(k)} * x_{i-1}, \dots, x_j^{(k)} * x_{i-1}, x_j * x_{i-1}, y, x_{j+1} * x_{i-1}, \dots, x_{i-2} * x_{i-1}, x_i, \dots, x_n) \\ &\quad - (x_j * x_{i-1}, \dots, x_j * x_{i-1}, x_j^{(k)} * x_{i-1}, y, x_{j+1} * x_{i-1}, \dots, x_{i-2} * x_{i-1}, x_i, \dots, x_n)\}. \end{aligned}$$

On the other hand, if $i \leq j$, then

$$G_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} \{(x_{j+1}^{(k)}, \dots, x_{j+1}^{(k)}, x_{j+1}, y, x_{j+2}, \dots, x_n) - (x_{j+1}, \dots, x_{j+1}, x_{j+1}^{(k)}, y, x_{j+2}, \dots, x_n)\},$$

so that the formula above does not depend on the binary operation $*$, and $G_{n-1}^j(d_i^{(*)0} - d_i^{(*)}) = 0$.

If $j + 1 \leq i$, then

$$\begin{aligned} G_{n-1}^j d_i^{(*)}(\mathbf{x}) &= \sum_{k=1}^m \sum_{y \in X} \{((x_j * x_i)^{(k)}, \dots, (x_j * x_i)^{(k)}, x_j * x_i, y, x_{j+1} * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n) \\ &\quad - (x_j * x_i, \dots, x_j * x_i, (x_j * x_i)^{(k)}, y, x_{j+1} * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)\}. \end{aligned}$$

Notice that $d_{i+1}^{(*)0} G_n^j = G_{n-1}^j d_i^{(*)0}$ and by Lemma 2.3 (2) $d_{i+1}^{(*)} G_n^j = G_{n-1}^j d_i^{(*)}$ if $j + 1 \leq i \leq n$.

Hence we have the following equality:

$$\partial_{n+1} G_n^j(\mathbf{x}) + G_{n-1}^j \partial_n(\mathbf{x}) = (-1)^{j+1} (|X| - m)(g_1^j(\mathbf{x}) - g_0^j(\mathbf{x})),$$

that means $(|X| - m)g_1^j$ and $(|X| - m)g_0^j$ are chain homotopic for each $1 \leq j \leq n$.

We next consider a chain homotopy $F_n^j : C_n^R(X) \rightarrow C_{n+1}^R(X)$ defined by

$$F_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} \{(x_j, \dots, x_j, y, x_j^{(k)}, x_{j+1}, \dots, x_n) - (x_j^{(k)}, \dots, x_j^{(k)}, y, x_j^{(k)}, x_{j+1}, \dots, x_n)\}$$

for $2 \leq j \leq n$. Then by a similar calculation as above, we have the following equality:

$$\partial_{n+1} F_n^j(\mathbf{x}) + F_{n-1}^j \partial_n(\mathbf{x}) = (-1)^j (|X| - m)(g_0^j(\mathbf{x}) - g_2^j(\mathbf{x})),$$

hence $(|X| - m)g_0^j$ is chain homotopic to $(|X| - m)g_2^j$ for each $2 \leq j \leq n$.

Therefore, we obtain the following sequence of chain homotopic chain maps

$$(|X| - m)g_1^j \simeq (|X| - m)g_0^j \simeq (|X| - m)g_2^j$$

for each $1 \leq j \leq n$. □

We now study annihilation of rack and quandle homology groups of m -almost quasigroup quandles under the same assumption as in Lemma 3.1 that every stabilizer set S_a is a trivial subquandle of X .

Theorem 3.2. *Let X be a finite m -almost quasigroup quandle. Suppose that for every $a \in X$ the stabilizer set S_a is a trivial subquandle of X . Then the torsion subgroup of $H_n^R(X)$ is annihilated by $m(\text{lcm}(|X|, |X| - m))$.*

Proof. If X is a trivial quandle, i.e. $m = |X|$, then we are done because the rack homology group of a trivial quandle does not have any torsion elements. Assume now that X is a nontrivial quandle. Then by Lemma 2.3 (4), the quandle X is connected. For each $j \in \{1, 2, \dots, n\}$, we define the symmetrizer chain map $g_s^j : C_n^R(X) \rightarrow C_n^R(X)$ by $g_s^j(\mathbf{x}) = \sum_{y \in X} (y, \dots, y, x_{j+1}, \dots, x_n)$.

We first prove that $mg_s^j - |X|g_2^j$ for $j \in \{1, \dots, n\}$ and $mg_s^{j-1} - |X|g_1^j$ for $j \in \{2, \dots, n\}$ are null-homotopic by using chain homotopies $D_n^j, E_n^j : C_n^R(X) \rightarrow C_{n+1}^R(X)$, respectively, given by

$$D_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} (x_j^{(k)}, \dots, x_j^{(k)}, x_j^{(k)}, y, x_{j+1}, \dots, x_n) \text{ for } 1 \leq j \leq n,$$

$$E_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} (x_j^{(k)}, \dots, x_j^{(k)}, y, x_j, x_{j+1}, \dots, x_n) \text{ for } 2 \leq j \leq n.$$

When $i \leq j$, by the idempotence condition of a quandle we obtain the following equation:

$$d_i^{(*)} D_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} (x_j^{(k)}, \dots, x_j^{(k)}, y, x_{j+1}, \dots, x_n).$$

Since the above formula does not depend on the binary operation $*$, $(d_i^{(*)0} - d_i^{(*)})D_n^j = 0$.

We check that $\sum_{y \in X} (x_j^{(k)} * y) = m(x_j^{(k)}) + \sum_{y \in X \setminus S_{x_j}} (y)$ because the stabilizer set S_{x_j} is trivial and

X satisfies the m -almost quasigroup property so that $\sum_{k=1}^m \sum_{y \in X} (x_j^{(k)} * y) = m \sum_{y \in X} (y)$.

Thus, if $i = j + 1$, then

$$d_i^{(*)} D_n^j(\mathbf{x}) = m \sum_{y \in X} (y, \dots, y, x_{j+1}, \dots, x_n)$$

therefore, $(d_i^{(*)0} - d_i^{(*)}) D_n^j = |X| g_2^j - m g_s^j$.

Finally, if $j + 2 \leq i \leq n + 1$, then we obtain $\sum_{y \in X} (y * x_{i-1}) = \sum_{y \in X} (y)$ from the invertibility condition of a quandle so that

$$d_i^{(*)} D_n^j(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} (x_j^{(k)} * x_{i-1}, \dots, x_j^{(k)} * x_{i-1}, y, x_{j+1} * x_{i-1}, \dots, x_{i-2} * x_{i-1}, x_i, \dots, x_n).$$

On the other hand, when $i \leq j$ we have the following equation:

$$D_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} (x_{j+1}^{(k)}, \dots, x_{j+1}^{(k)}, y, x_{j+2}, \dots, x_n),$$

hence $D_{n-1}^j (d_i^{(*)0} - d_i^{(*)}) = 0$ because the formula above does not depend on the operation $*$.

If $j + 1 \leq i$, then we have

$$D_{n-1}^j d_i^{(*)}(\mathbf{x}) = \sum_{k=1}^m \sum_{y \in X} ((x_j * x_i)^{(k)}, \dots, (x_j * x_i)^{(k)}, y, x_{j+1} * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n).$$

Note that if $j + 1 \leq i \leq n$, then $d_{i+1}^{(*)0} D_n^j = D_{n-1}^j d_i^{(*)0}$ and by Lemma 2.3 (2) $d_{i+1}^{(*)} D_n^j = D_{n-1}^j d_i^{(*)}$.

Finally, we obtain the following equality:

$$\partial_{n+1} D_n^j(\mathbf{x}) + D_{n-1}^j \partial_n(\mathbf{x}) = (-1)^{j+1} (|X| g_2^j(\mathbf{x}) - m g_s^j(\mathbf{x})),$$

which means the chain maps $|X| g_2^j$ and $m g_s^j$ are chain homotopic for each $1 \leq j \leq n$.

By using the chain homotopy $E_n^j : C_n^R(X) \rightarrow C_{n+1}^R(X)$ and a similar calculation as above, we also have the following equality:

$$\partial_{n+1} E_n^j(\mathbf{x}) + E_{n-1}^j \partial_n(\mathbf{x}) = (-1)^j (|X| g_1^j(\mathbf{x}) - m g_s^{j-1}(\mathbf{x})),$$

so that the chain maps $|X| g_1^j$ and $m g_s^{j-1}$ are chain homotopic for each $2 \leq j \leq n$.

Then from the above calculations and by Lemma 3.1, we have a sequence of chain homotopic chain maps

$$N \text{Id} \simeq \frac{N}{m} g_2^1 \simeq \frac{N}{|X|} g_s^1 \simeq \frac{N}{m} g_1^2 \simeq \frac{N}{m} g_2^2 \simeq \frac{N}{|X|} g_s^2 \simeq \dots \simeq \frac{N}{m} g_1^n \simeq \frac{N}{m} g_2^n \simeq \frac{N}{|X|} g_s^n$$

where $N = m(\text{lcm}(|X|, |X| - m))$. We therefore obtain the same induced homomorphisms

$N \text{Id} = (\frac{N}{|X|} g_s^n)_* : H_n^R(X) \rightarrow H_n^R(X)$. Recall that since X is connected, the free part of the n th

rack homology group of X is \mathbb{Z} generated by (y, \dots, y) for $y \in X$. Hence $N\text{tor}(H_n^R(X)) = 0$ for any dimension n . \square

Corollary 3.3. *Let X be as in Theorem 3.2. If X is nontrivial, then the reduced quandle homology of X is annihilated by $N = m(\text{lcm}(|X|, |X| - m))$, i.e. $N\tilde{H}_n^Q(X) = 0$.*

Proof. The rack homology of a quandle splits into degenerate homology and quandle homology, i.e. $H_n^R(X) = H_n^D(X) \oplus H_n^Q(X)$ (see [L-N]), hence the torsion of $H_n^Q(X)$ is annihilated by N by Theorem 3.2. Furthermore, since X is finite and nontrivial, X is a finite connected quandle by Lemma 2.3 (4). Then $\text{rank}(H_n^Q(X)) = 0$ for $n \geq 2$ and $\text{rank}(H_1^Q(X)) = 1$. Therefore, the reduced quandle homology of X is a torsion group annihilated by N . \square

Corollary 3.4 is immediate from Lemma 2.3 (3) and Theorem 3.2.

Corollary 3.4. *Let X be a finite m -almost quasigroup quandle where $m \leq 3$. Then the torsion subgroup of $H_n^R(X)$ is annihilated by $N = m(\text{lcm}(|X|, |X| - m))$.*

The connected quandle $QS(6)$ (which is not a quasigroup quandle) is used to show that for any connected quandles the torsion subgroup of its rack and quandle homology can not be annihilated by its order in general. However, we obtain the best solution for the quandle $QS(6)$ from Theorem 3.2 and Corollary 3.3.

Example 3.5. The least upper bound for orders of the torsion elements in $H_n^R(QS(6))$ is $|\text{Inn}(QS(6))|$.

For every dimension n , in particular, $|\text{Inn}(QS(6))|\tilde{H}_n^Q(QS(6)) = 0$.

Since $QS(6)$ is a 2-almost quasigroup quandle of order 6, 24 is an upper bound for orders of the torsion elements in $H_n^R(QS(6))$ by Theorem 3.2. Then $\text{Inn}(QS(6)) \cong S_4$ and $H_3^Q(QS(6)) = \mathbb{Z}_{24}$ (see [CKS-1]) implies that $|\text{Inn}(QS(6))| = |S_4| = 24$ is the smallest number which annihilates $\text{tor}H_n^R(QS(6))$.

Remark 3.6. *The order of a quandle inner automorphism group is not always the least upper bound for orders of the torsion elements in the rack or quandle homology groups of the quandle. For example, the 3-almost quasigroup quandle of order 12 (Rig quandle $Q(12, 10)$) given by the Table 3 (see [Cla]) has $\text{Inn}(Q(12, 10))$ of order 216 while Theorem 3.2 shows that 108 annihilates the torsion of homology.*

Example 3.7. According to [Ven] there are seven connected quandles of 15 elements. Five of them are quasigroup quandles while the Rig quandle $Q(15, 2)$ is a 3-almost quasigroup quandle and the Rig quandle $Q(15, 7)$ is a 7-almost quasigroup quandle. Furthermore, by Corollary 3.4 $\text{tor}H_n(Q(15, 2))$ is annihilated by $3(\text{lcm}(15, 15 - 3)) = 180$. We have⁴ $H_2^Q(Q(15, 2)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_3^Q(Q(15, 2)) = \mathbb{Z}_2 \oplus \mathbb{Z}_{30}$, therefore it is not annihilated by any power of $|X| = 15$. This is possible because the Rig quandle $Q(15, 2)$ is not a homogeneous quandle, so the result in [L-N] does not apply. The group $\text{Inn}(Q(15, 2))$ has 60 elements, so it annihilates $H_2^Q(Q(15, 2))$ and $H_3^Q(Q(15, 2))$.

4. FUTURE RESEARCH

We plan to work on the problem of when the order of the inner automorphism group of a quandle annihilates the torsion of its homology, i.e. $|\text{Inn}(X)|\text{tor}H_n(X) = 0$. We have several examples for which it is the case:

⁴We would like to thank L. Vendramin for computing the homology $H_3^R(Q(15, 2)) = \mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_{30}$.

TABLE 3. A 3-almost quasigroup quandle of order 12; $Q(12, 10)$

*	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	12	11	10	5	4	6	9	7	8
2	2	2	2	11	10	12	6	5	4	8	9	7
3	3	3	3	10	12	11	4	6	5	7	8	9
4	8	9	7	4	4	4	10	12	11	3	2	1
5	7	8	9	5	5	5	11	10	12	2	1	3
6	9	7	8	6	6	6	12	11	10	1	3	2
7	11	12	10	3	1	2	7	7	7	4	5	6
8	12	10	11	1	2	3	8	8	8	5	6	4
9	10	11	12	2	3	1	9	9	9	6	4	5
10	6	5	4	7	8	9	3	2	1	10	10	10
11	5	4	6	9	7	8	1	3	2	11	11	11
12	4	6	5	8	9	7	2	1	3	12	12	12

Example 4.1. (1) Finite quasigroup quandles.

For a finite quasigroup quandle X , $|X|$ annihilates $\text{tor}H_n(X)$ by Theorem 1.8 and Corollary 1.9. Furthermore, since X embeds in $\text{Inn}(X)$, we know that $|X|$ divides $|\text{Inn}(X)|$ by the orbit-stabilizer theorem. Therefore $|\text{Inn}(X)|\text{tor}H_n(X) = 0$. In particular, for odd k , $\text{tor}H_n(R_k)$ is annihilated by k because the dihedral quandle R_k is a quasiguop quandle.

(2) Rig quandles $Q(6, 1)$, $Q(6, 2)$, $Q(12, 8)$, and $Q(12, 9)$.

Since $Q(6, 1)$ and $Q(6, 2)$ are 2-almost quasigroup quandles and $Q(12, 8)$ and $Q(12, 9)$ are 4-almost quasigroup quandles, we see that $\text{tor}H_n(Q(6, 1))$ and $\text{tor}H_n(Q(6, 2))$ are annihilated by 24 and $\text{tor}H_n(Q(12, 8))$ and $\text{tor}H_n(Q(12, 9))$ are annihilated by 96 by Theorem 3.2 and Corollary 3.3. Moreover, $\text{Inn}(Q(6, 1))$ and $\text{Inn}(Q(6, 2))$ are isomorphic to the symmetric group S_4 of order 24 and $|\text{Inn}(Q(12, 8))| = 96 = |\text{Inn}(Q(12, 9))|$ by [Cla]. Therefore, in all cases described above the order of the inner automorphism group of each quandle annihilates the torsion of its homology.

It is still an open problem whether the order of the quandle inner automorphism group of a quandle annihilates the torsion subgroup of its rack and quandle homology for every finite connected quandle. We can make this problem more general, i.e. we ask whether for any finite quandle X , $\text{tor}H_n(X)$ is annihilated by $|\text{Inn}(X)|$. So far, we can show that if k is odd, then k annihilates the torsion of rack and quandle homology of R_{2k} (compare Conjecture 14 in [N-P-2]). Notice that R_{2k} is a non-connected quandle and $\text{Inn}(R_{2k})$ is isomorphic to the dihedral group D_k of order $2k$.

The smallest connected quandle for which our method does not work is the Rig quandle $Q(8, 1)$ which is the abelian extension of the Alexander quandle $Q(4, 1) = \mathbb{Z}_2[t]/(t^2 + t + 1)$. The order of the quandle inner automorphism group of $Q(8, 1)$ is 24; $H_2^Q(Q(8, 1)) = 0$ and $H_3^Q(Q(8, 1)) = \mathbb{Z}_8$.

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